Modified Laplace transformation method at finite temperature: application to infra-red problems of N component ϕ^4 theory

H. YAMADA
e-mail: yamadah@cc.it-chiba.ac.jp
Natural Science, General Education
Chiba Institute of Technology
Shibazono 2-1-1
Narashino, Chiba 275
Japan

February 1, 2008

ABSTRACT

Modified Laplace transformation method is applied to N component ϕ^4 theory and the finite temperature problem in the massless limit is re-examined in the large N limit. We perform perturbation expansion of the dressed thermal mass in the massive case to several orders and try the massless approximation with the help of modified Laplace transformation. The contribution with fractional power of the coupling constant is recovered from the truncated massive series. The use of inverse Laplace transformation with respect to the mass square is crucial in evaluating the coefficients of fractional power terms.

1 Introduction

It is well known that the conventional perturbation theory for bosons breaks down when the temperature is sufficiently high compared to the mass scale of the theory under consideration[1]. The origin is easily seen in the massless limit as the emergence of infra-red divergences of Feynman diagrams. This is linked to the fact that the perturbative infra-red structure becomes worse at finite temperature. For example, the boson propagator shows the behavior more singular at small momentum region than at zero temperature, and the temperature correction remains finite for the self-energy while at zero temperature it vanishes at zero mass in dimensional regularization. The nonvanishing of mass correction activates the hidden infra-red divergence of sub-diagrams coupled to the tadpole, and thus the divergence manifests itself in Feynman amplitudes.

The presence of the thermal mass means that there would be no infra-red problem if one can incorporate such effect from the outset of perturbation expansion. It also means, as well known, that the ordinary perturbation expansion in powers of the coupling constant breaks down. For example, in a single component ϕ^4 theory with $\lambda\phi^4$ interaction, the pressure P is given at temperature T $(= 1/\beta)$ by

$$P = T^4\Big[\frac{\pi^2}{90} - \frac{1}{48}\lambda + \frac{1}{12\pi}\lambda^{3/2} + O(\lambda^2)\Big]. \tag{1}$$

The presence of the fractional power of the coupling constant λ exhibits the break down of perturbation expansion.

To obtain the amplitude free from the infra-red singularity, one may sum up a set of Feynman diagrams to all orders. Similar but more systematic approach would be to shift the perturbative vacuum such that the fields are thermally screened at the tree level[2,3,4,5]. This program is implemented by adding the thermal mass to be induced by the loop correction to the free part and subtracting from the interaction part. Although this is

physically appealing, it is not effective in the study of magnetic part of gluon self-energy, because the one-loop contribution is absent and higher order diagrams are plagued with the infra-red singularity. This is a reason why we look for another expansion scheme.

Recently a new perturbative scheme was proposed[6,7] and it seems to be a convenient scheme to deal with the infra-red problem at finite temperature. The scheme is based on the use of the Laplace transformation which is modified as to fit the perturbative framework. In the case of anharmonic oscillator, the infra-red problem at zero temperature is well controlled by the modified Laplace transformation (MLT) method[7]. The purpose of the present paper is to apply the method to the finite temperature perturbation theory of N component $\lambda\phi^4$ theory at $D = 4$ [8]. To check our result we confine ourselves with the leading order of $1/N$ expansion. We consider the self-energy and investigate how the non-analytic structure in the coupling constant similar to (1) can be recovered from massive perturbation expansion which gives a simple power series in the coupling constant. We do not sum up Feynman diagrams to all orders, but just deal with the truncated series. Nevertheless we will show that first several coefficients of the fractional power terms can be obtained from the truncated perturbative series.

2 Survey of the modified Laplace transformation method

For the self-containedness we here survey the basic points of MLT method[6,7]. Since we confine ourselves with the approximation of a massless theory, our presentation emphasizes the relevant respects only.

Let $f(\sigma)$ be a physical function and σ be monotonic with respect to the mass. Then the massless limit is equivalent with the limit, $\sigma \to 0$. Suppose that $f(\sigma)$ has perturbative expansion in $1/\sigma$ and is given at the perturbative order k as $f_k(\sigma) = \sigma^p \sum_{n=0}^{k} a_n/\sigma^k$ where p is some constant. Then the "infra-red problem" occurs when we ask the behavior of $f(\sigma)$

for small σ. This is the situation we encounter at finite temperature ϕ^4 theory. Since we cannot take the massless limit in $f_k(\sigma)$, we may fix σ at some small value where the lower limit of perturbation series is signaled. In this approximation scheme, it is convenient to consider the inverse Laplace or Heaviside transform with respect to σ as we can see below.

Heaviside transform of $f(\sigma)$ is given by the Bromwich integral,

$$\hat{f}(x) = \int_{s-i\infty}^{s+i\infty} \frac{d\sigma}{2\pi i} \exp(\sigma x) \frac{1}{\sigma} f(\sigma), \tag{2}$$

where the parameter s represents the location of the vertical contour. The contour of integration should be placed on the right side to all the possible poles and the cut of $f(\sigma)/\sigma$. Then, if $x < 0$, the contour may be closed into the right half circle and $\hat{f}(x)$ is found to vanish. As well known, $\hat{f}(x)$ is the kernel of Laplace transformation of $f(\sigma)$. We have

$$f(\sigma) = \sigma \int_{-\infty}^{\infty} dx \exp(-\sigma x) \hat{f}(x). \tag{3}$$

Since $\hat{f}(x) = 0$ when $x < 0$, the integration range reduces to $[0, \infty)$. However $\hat{f}(x)$ involves the step and δ functions and it is generally convenient to keep the range as $(-\infty, +\infty)$ to handle integration easily.

In our massless approximation, we begin with the observation that

$$\lim_{\sigma \to 0} f(\sigma) = \lim_{x \to \infty} \hat{f}(x), \tag{4}$$

where we assumed the existence of the limits. This ensures us that $\hat{f}(x)$ can be equally used for the massless approximation. Let us denote the Heaviside transform of $f_k(\sigma)$ as $\hat{f}_k(x)$. Although we cannot take the massless limit for $\hat{f}_k(x)$, we may approximate $\hat{f}(\infty)$ by fixing x some large value, x^*, in $\hat{f}_k(x)$. Then we note that, in some cases including our finite temperature problem, the transformed function has an advantage that the convergence radius is larger than $f(\sigma)$. For example, when $f = \sum a_n/\sigma^n$, $\hat{f} = \sum (a_n/n!) x^n$. Therefore,

if $\lim_{x\to\infty}\hat{f}(x)$ converges, then the transformed function gives us more chance to know the massless value by increasing the order of expansion.

To summarize, calculating $\hat{f}_k(x)$ from (2), we approximate $f(0)$ by $\hat{f}_k(x^*)$. The explicit way of fixing x^* will be discussed in the next section.

This approach can be naturally extended to the calculation of the massive case[7]. Let $f_k(\sigma, x^*)$ be the k-th order approximation of $f(\sigma)$. Then $f_k(\sigma, x^*)$ is given by

$$f_k(\sigma, x^*) = \exp(-\sigma x^*)\hat{f}_k(x^*) + \sigma \int_{-\infty}^{x^*} dx \exp(-\sigma x)\hat{f}_k(x). \tag{5}$$

The right-hand-side defines our modification of Laplace transformation. In addition to the naive cut-off term (the second term), we are led to add the first term. This term is critical in approximating $f(\sigma)$ for small σ.

In performing Heaviside transformation, the following formula is useful;

$$\sigma^p \to \frac{x^{-p}}{\Gamma(1-p)}\theta(x) \quad (p \le 1), \tag{6}$$

where $\theta(x)$ denotes the step function ($\theta(x) = 0$ for $x < 0$ and 1 for $x > 0$). For larger p one can obtain the transform by using

$$\sigma f(\sigma) \to \frac{\partial \hat{f}(x)}{\partial x}. \tag{7}$$

For example we have

$$\sigma^{1+p} \to -p\frac{x^{-1-p}}{\Gamma(1-p)}\theta(x) + \frac{x^{-p}}{\Gamma(1-p)}\delta(x). \tag{8}$$

In this way we find

$$\sigma^n \to \begin{cases} 1 & (n = 0) \\ 0 & (n = 1, 2, 3, \cdots) \end{cases} \tag{9}$$

where we have omitted the θ and δ functions. We note that for fractional p the transform does not vanish.

3 Application of MLT method

We work with the imaginary time formalism of finite temperature field theory. The model we consider is defined by[8]:

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda\mu^{2\epsilon}}{4N}\phi^4 + \mathcal{L}_{E,c}, \tag{10}$$

where $\phi^2 = \sum_{n=1}^{N} \phi_n^2$ and $\mathcal{L}_{E,c}$ denotes the collection of counter terms. We use dimensional regularization at $D = 4 - 2\epsilon$ with MS prescription. Before turning to detailed calculation, we spend some words on our basic strategy.

We write the massive perturbation expansion of the thermal mass to order k as

$$M^2 = m^2 + \sum_{n=1}^{k} c_n(m) g^n. \tag{11}$$

Since we are interested in the small mass behavior of massive perturbation theory, we expand each loop correction in the mass. This is equivalent to use the high temperature expansion. For example, the one-loop contribution to the thermal mass gives

$$\begin{aligned} c_1 &= g\Big[\frac{8\zeta(2)}{\beta^2} - \frac{4\pi m}{\beta} - tm^2 + \frac{8\pi^{3/2}}{\beta^2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\Big(\frac{\beta^2 m^2}{4\pi^2}\Big)^n \zeta(2n-1)\Gamma(n-1/2)\Big] \\ &= g\Big[\frac{8\zeta(2)}{\beta^2} - \frac{4\pi m}{\beta} + R_{even}(m^2)\Big], \end{aligned} \tag{12}$$

where

$$g = \frac{\lambda}{(4\pi)^2}, \quad t = \gamma + \log\frac{\beta^2\mu^2}{4\pi}. \tag{13}$$

The term linear in m is the crucial one in this correction. It is in a sense non-perturbative, since the mass enters in the Lagrangian as the square of the mass, m^2. Actually all the infra-red divergent behavior of sub-diagrams with many ϕ legs comes from that term by the differentiation with respect to m^2.

There is a subtlety in choosing the transformation variable σ. For instance we can enlarge the convergence radius of the thermal mass, $M^2(m)$, in both cases where $\sigma = m$

and $\sigma = m^2$. In this paper, we choose the choice, $\sigma = m^2$, according to the successful choice in the anharmonic oscillator[7].

Heaviside transform of $c_1(m)$, $\hat{c}_1(x)$, is then given by

$$\hat{c}_1(x) = g\Big[\frac{8\zeta(2)}{\beta^2} - \frac{4\pi}{\beta\sqrt{x}\Gamma(1/2)} + \hat{R}_{even}(x)\Big], \tag{14}$$

where $\hat{R}_{even}(x)$ denotes the Heaviside transform of $R_{even}(m^2)$. Since any term in $R_{even}(m^2)$ is an even positive power of m, any finite sum involved gives zero (see (9)). Although this does not necessarily mean that $\hat{R}_{even}(x)$ is zero, we may safely neglect it when we focus on the massless approximation. On the other hand, while the second term, $4\pi/\beta\sqrt{x}\Gamma(1/2)$, tends to zero when $x \to \infty$, we keep it because it gives non-negligible contribution at finite perturbative order where x is fixed to some finite value in the massless approximation. To summarize, we keep terms as the transformation suggests: Terms needed are of negative, zero and odd powers of m (which corresponds to positive, zero and fractional powers of x).

Before performing our approximate calculation to the full several loops, let us illustrate in the next subsection how the leading fractional term of order $g^{3/2}$ comes out in our approach.

3.1 Calculation of $g^{3/2}$ term

It is well known that the leading fractional term of order $g^{3/2}$ comes from the series of Feynman diagrams shown in Fig.1, each of which gives most dominant contribution for small m at each order in g. Let us first focus on the dominant graphs and try to approximate the leading fractional term.

The diagrams shown in Fig.1 gives:

$$M^2_{dominant}(m) = \frac{8\zeta(2)}{\beta^2}\Big[g\Big(1-\frac{3\beta m}{\pi}\Big)-\frac{2\pi}{\beta m}g^2+\frac{8\zeta(2)\pi}{\beta^3 m^3}\frac{g^3}{2!}-\frac{(8\zeta(2))^2\pi}{\beta^5 m^5}\frac{1\cdot 3}{3!2}g^4$$

$$+ \quad \frac{(8\zeta(2))^3\pi}{\beta^7 m^7}\frac{1\cdot 3\cdot 5}{4!2^2}g^5+\cdots\Big] + (\text{sub dominant terms}), \tag{15}$$

where (sub dominant terms) denotes the sub-dominant pieces in the small m expansion. Hereafter we omit those sub-dominant terms. We have kept the next-to-the leading term (the second one in (15)) in the one-loop contribution. This is because the leading contribution of each higher order diagram comes from this term and the power counting of higher terms with respect to m suggests that it is the first term in the dominant series. From (6) the Heaviside transform of $M^2_{dominant}$ with respect to m^2 is found to give

$$\begin{aligned} \hat{M}^2_{dominant}(x) &= \frac{8\zeta(2)}{\beta^2}\Big[g\Big(1-\frac{3\beta}{\pi\sqrt{x}\Gamma(1/2)}\Big)-\frac{2\pi\sqrt{x}}{\beta\Gamma(3/2)}g^2+\frac{8\zeta(2)\pi x^{3/2}}{\Gamma(5/2)\beta^3}\frac{g^3}{2!} \\ &- \frac{(8\zeta(2))^2\pi x^{5/2}}{\Gamma(7/2)\beta^5}\frac{1\cdot 3}{3!2}g^4+\cdots\Big]. \end{aligned} \tag{16}$$

Nontrivial result is obtained from the 2-loop order. At 2-loop order, we have

$$\hat{M}^2_{dominant}(x)=\frac{8\zeta(2)}{\beta^2}\Big[g\Big(1-\frac{3\beta}{\pi\sqrt{x}\Gamma(1/2)}\Big)-\frac{2\pi\sqrt{x}}{\Gamma(3/2)\beta}g^2\Big]. \tag{17}$$

The function $\hat{M}^2_{dominant}(x)$ behaves as shown in Fig.2 and the limitation of the result is seen, for example for $g=0.05$, at $x/\beta^2\sim 1.52$. When x/β^2 exceeds 1.52, the highest term in (17) dominates over and therefore the result (17) loses its validity for larger x. Thus, the upper limit of perturbative region may be given at the stationary point of $\hat{M}^2_{dominant}(x)$ and it is specified by

$$\frac{\partial \hat{M}^2_{dominant}(x)}{\partial x}=\frac{8\zeta(2)}{\beta^2}\Big[\frac{3g\beta}{2\pi\Gamma(1/2)x^{3/2}}-\frac{\pi g^2}{\Gamma(3/2)\beta x^{1/2}}\Big]=0. \tag{18}$$

We note that the first term in (17), which is independent of x, does not enter the stationarity condition (18). The solution of (18) is given by

$$x^*=\frac{3\beta^2}{4\pi^2 g}. \tag{19}$$

Note that x^* is proportional to g^{-1}. Hence the fractional power in x produces the fractional power of g. This is the mechanism to generate the fractional power of g which generally

holds in our approach. By substituting x^* into $\hat{M}^2_{dominant}(x)$ we obtain the approximation of $M^2_{dominant}(m=0)$. The result reads

$$\hat{M}^2_{dominant}(x^*) = \frac{8\zeta(2)}{\beta^2}\Big[g - \frac{4\sqrt{3}}{\sqrt{\pi}} g^{3/2}\Big]. \tag{20}$$

Thus, we have recovered the $g^{3/2}$ term from the truncated power series in g. Since the exact result reads $M^2(m=0) = 8\zeta(2)/\beta^2\Big[g - 2\sqrt{3}g^{3/2} + O(g^2)\Big]$, the first term in (20) is exact and the error in $O(g^{3/2})$ is about 13 percents. This is satisfactory as the lowest order approximation.

Now we investigate the effect of multi-loop dominant graphs. Since the stationarity condition always gives x^* as $\beta^2/(\zeta(2)g)$ times the number, it is convenient to define

$$\sigma = \frac{\beta^2 m^2}{8\zeta(2)g}. \tag{21}$$

and consider the transformation with respect to σ. We note that m^2/g is renormalization group (RG) invariant and therefore σ is. In terms of σ, (15) is written as

$$M^2_{dominant} = \frac{8\zeta(2)}{\beta^2}\Big[g - \sqrt{3}g^{3/2}A(\sigma)\Big], \tag{22}$$

where

$$A(\sigma) = \sqrt{\sigma}\Big[2 + \frac{1}{\sigma} - \frac{1}{\sigma^2}\frac{1}{2!2} + \frac{1}{\sigma^3}\frac{1\cdot 3}{3!2^2} - \frac{1}{\sigma^4}\frac{1\cdot 3\cdot 5}{4!2^3} + \cdots\Big] = \sqrt{\sigma}\Big[2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{\sigma^{n+1}}\frac{(2n-1)!!}{(n+1)!2^n}\Big]. \tag{23}$$

Let us truncate (23) to the $k+2$ loop level. The truncated series is given by

$$A_k(\sigma) = 2\sqrt{\sigma} + \sum_{n=0}^{k}\frac{(-1)^n}{\sigma^{n+1/2}}\frac{(2n-1)!!}{(n+1)!2^n} \tag{24}$$

The corresponding Heaviside function $\hat{A}_k(x)$ is given by

$$\hat{A}_k(x) = \frac{2}{\Gamma(1/2)x^{1/2}} + \sum_{n=0}^{k}\frac{(-1)^n(2n-1)!!}{(n+1)!2^n\Gamma(n+3/2)}x^{n+1/2}. \tag{25}$$

Here x is the conjugate variable of σ defined by (21) and should not be taken as the previous x appeared in (16), (17), (18) and (19). We emphasize that, while the convergence radius

of $A(\sigma)$ is 1, that of $\hat{A}(x)$ is infinite. As mentioned in the introduction, this is the essential advantage in dealing with $\hat{A}_k(x)$ rather than $A_k(\sigma)$. Our task is to approximate $A(0)$, which is equal with 2, by $\hat{A}_k(x)$ where x may be optimized by the stationarity requirement. The stationarity condition to fix x leads to $\partial\hat{A}_k(x)/\partial x = 0$ and the solution is found at even orders. By substituting the solution into $\hat{A}_k(x)$, we obtain the following approximation for $A(0) = 2$;

$$\begin{aligned} 2-loop &: \quad 2.25676 \quad (x^* = 1), \\ 4-loop &: \quad 2.06054 \quad (x^* = 1.596), \\ 6-loop &: \quad 2.02096 \quad (x^* = 2.181), \\ 8-loop &: \quad 2.00835 \quad (x^* = 2.759), \\ 10-loop &: \quad 2.00358 \quad (x^* = 3.334). \end{aligned} \tag{26}$$

Thus, the sequence gives the good approximation of $A(0)$ and $g^{3/2}$ term. Finally we point out that the omitting of the first term in $A_k(\sigma)$ (see (24)) gives poor result and its necessity is confirmed.

3.2 Approximation via full loop expansion to 10-loops

The previous procedure can be applied to the full loop corrections of the dressed thermal mass.

We first sum up the logarithmic corrections by fixing the value of t. Let us fix t simply by

$$t = \gamma + \log\frac{\beta^2\mu^2}{4\pi} = 0. \tag{27}$$

This gives $\mu^2 = 4\pi e^{-\gamma}/\beta^2$ and the coupling constant g is given by

$$g = \left[\log\frac{\beta^2\Lambda^2}{4\pi} + \gamma\right]^{-1} \equiv g_\beta, \tag{28}$$

where Λ denotes the intrinsic scale in MS scheme. If one wishes to recover the ordinary logarithmic terms, one can do it by expanding g_β as $g_\beta = g(\mu)\Big(1 + tg(\mu) + t^2 g^2(\mu) + \cdots\Big)$. In this case, RG invariance will be satisfied to the same level of ordinary perturbative expansion.

From 2-loop level, the terms with odd power of m appear to all orders in the high temperature expansion. For instance

$$\begin{aligned} \frac{\beta^2}{8\zeta(2)} c_2 &= g_\beta^2 \Big(6 - \frac{2\pi}{3\beta m} - \frac{15\beta^3 m^3}{16\pi^3}\zeta(3) + \frac{21\beta^5 m^5}{128\pi^5}\zeta(5) + \cdots\Big) \\ &= 6g_\beta^2 - \sqrt{3} g_\beta^{3/2}\Big[\frac{1}{\sqrt{\sigma}} + \frac{5}{6} g_\beta^2 \zeta(3)\sigma^{3/2} - \frac{7}{36} g_\beta^3 \zeta(5)\sigma^{5/2} + \cdots\Big], \end{aligned} \tag{29}$$

where $\cdots$ denotes the higher order terms in m and even powers of m is fully omitted. The terms with positive and odd powers of m survive under the Heaviside transformation with respect to m^2. Hence, although those terms go to zero as $\sigma \to 0$, they are needed at finite order massless approximation. We note that any term in the small mass expansion of c_2 is of g_β^2-order, but the higher order term in m in fact contributes to the higher order in g_β of massless approximant as suggested in (29).

In small mass or high-temperature expansion, m independent pieces (the first term in (12) and (29), for example) play a special role under our approach because those are not affected by our approximation scheme and go through to the final result. Those terms are collected as

$$\frac{8\zeta(2)}{\beta^2}\Big[g_\beta + 6g_\beta^2 + \frac{1}{6}\zeta(3) g_\beta^3 + O(g_\beta^4)\Big] \equiv M_g^2 \tag{30}$$

and give the purely power like part of $M^2(m=0)$. Actually the above result agrees with the exact result. On the other hand, the terms with fractional power of σ give rise to the fractional power (in g) part of the approximant of $M^2(m=0)$. At this stage, let us summarize the classification of terms responsible for the approximation of $M^2(m=0)$. The terms zero-th order in small mass expansion were gathered into M_g^2 and exactly gives

the integer power part of $M^2(m=0)$. The terms of odd power of m are gathered into $M^2_{\sqrt{g}}$ and contribute to the fractional power part of $M^2(m=0)$. Thus, neglecting terms of even positive powers of m (including the tree-level contribution), we can put

$$M^2(m) = M_g^2 + M^2_{\sqrt{g}}(m). \tag{31}$$

Now we explicitly carry out the approximation of the massless value of $M^2_{\sqrt{g}}$. We have calculated the radiative corrections c_n up to 10-loops and classified the result in powers of g_β. Then we can write

$$\begin{aligned} M^2_{\sqrt{g}} &= -\sqrt{3} g_\beta^{3/2} f(\sigma, g_\beta) \\ &= -\sqrt{3} g_\beta^{3/2} \Big[A(\sigma) + g_\beta B(\sigma) + g_\beta^2 C(\sigma) + O(g_\beta^3) \Big]. \end{aligned} \tag{32}$$

We further classified coefficient functions, $A(\sigma), B(\sigma), \cdots$, based on ζ functions. The result is shown in appendix. We thus arrive at

$$\begin{aligned} M^2_{\sqrt{g}} &= -\sqrt{3} g_\beta^{3/2} \Big[2a + 3b g_\beta + \Big(\frac{5c_{\zeta(3)}}{6}\zeta(3) - \frac{9c}{4} \Big) g_\beta^2 + \Big(\frac{35 d_{\zeta(3)}}{4}\zeta(3) - \frac{7 d_{\zeta(5)}}{36}\zeta(5) + \frac{27}{8} d \Big) g_\beta^3 \\ &\quad + \Big(\frac{189 e_{\zeta(3)}}{16}\zeta(3) + \frac{7 e_{\zeta(3)^2}}{16}\zeta(3)^2 - \frac{35 e_{\zeta(5)}}{8}\zeta(5) + \frac{5 e_{\zeta(7)}}{96}\zeta(7) - \frac{405 e}{64} \Big) g_\beta^4 + O(g_\beta^5) \Big] \end{aligned} \tag{33}$$

Our massless approximation goes as in the previous subsection: We first perform the Heaviside transformation of $M^2(m)$ which gives $\hat{M}^2(x) = M_g^2 + \hat{M}^2_{\sqrt{g}}(x)$. Here $\hat{M}^2_{\sqrt{g}}(x)$ is given by the series same as (33) except that the coefficient functions $a(\sigma)$, $b(\sigma)$, $\cdots$ are now replaced by the Heaviside transforms, $\hat{a}(x), \hat{b}(x), \cdots$. Then we look for the stationary points of them to evaluate their massless values. The result is shown in Table 1.

We find that, at the 10-loop level, the first 5 coefficient functions, A, B, C, D and E are approximately evaluated. The coefficients a, b, c_ζ, d_ζ and e_ζ already reach to the sufficient accuracy at this level. The accuracy of c, d and e is however not so good. The reason is that $\hat{c}(x), \hat{d}(x)$ and $\hat{e}(x)$ start with $x^{3/2}, x^{5/2}$ and $x^{7/2}$ terms, respectively, and therefore need higher order contributions to roughly reach their values at $x=\infty$. From Table 1, we

obtain the following approximant of the fractional power part of $M^2(m=0)$,

$$M^2_{\sqrt{g}} = -\sqrt{3} g_\beta^{3/2} \Big[2.0036 + 2.9220 g_\beta - 0.6172 g_\beta^2 + 11.2082 g_\beta^3 + 8.7295 g_\beta^4 + O(g_\beta^5) \Big]. \quad (34)$$

The exact result reads,

$$M^2_{\sqrt{g}}(m=0) = -\sqrt{3} g_\beta^{3/2} \Big[2 + 3 g_\beta - 0.8075 g_\beta^2 + 13.6914 g_\beta^3 + 4.0193 g_\beta^4 + O(g_\beta^5) \Big]. \quad (35)$$

In view of (34) and (35), we can say that our approach can produce the fractional power part of the self-energy to the satisfactory level.

4 Conclusion

In the large N limit of N component ϕ^4 theory, we found the mechanism that how the integer and the fractional power terms in the massless theory are systematically classified and generated via massive perturbation theory. The fractional terms comes from the odd power terms of m in the high temperature expansion. To 10-loop order, the terms of order $g_\beta^{3/2}, g_\beta^{5/2}, g_\beta^{7/2}, g_\beta^{9/2}$ and $g_\beta^{11/2}$ were approximately calculated with the help of MLT method. Although only the two point function is considered in this paper, our approach would be directly applied to the thermodynamic functions, without referring to the mass correction.

In our approach, we can deal with all the fractional power terms on equal footing. We have not performed infinite sum which will be difficult in general. The characteristic features in our approach are that the evaluation of the massless values is based on the truncated sum which can be obtained for general theories and, even when the one-loop contribution vanishes at $m = 0$, our evaluation procedure works. Therefore we expect that our approach may be effective for more complicated theories. Long time ago, Linde observed that perturbative hot QCD breaks down at and above a certain perturbative order[9], since the magnetic gluon mass would be of order g^2T and then the infinite number of diagrams contribute to a single order in g (See also refs. 10,1). Our approach may

provide a new calculational framework toward the resolution of the problem. Although the calculation of higher order diagrams is tedious and involved, the fundamental problems issued by Linde become harmless.

Appendix

In this appendix we summarize our perturbative calculation. The function f is represented as

$$f(\sigma) = A(\sigma) + g_\beta B(\sigma) + g_\beta^2 C(\sigma) + g_\beta^3 D(\sigma) + g_\beta^4 E(\sigma) + O(g_\beta^4).$$

Each coefficient function is given to 10-loops as follows:

$$\begin{aligned}
A &= 2a(\sigma),\\
B &= 3b(\sigma),\\
C &= \frac{5\zeta(3)}{6}c_{\zeta(3)}(\sigma) - \frac{9}{4}c(\sigma),\\
D &= \frac{35}{4}\zeta(3)d_{\zeta(3)}(\sigma) - \frac{7\zeta(5)}{36}d_{\zeta(5)}(\sigma) + \frac{27}{8}d(\sigma),\\
E &= \frac{189\zeta(3)}{16}e_{\zeta(3)}(\sigma) + \frac{7\zeta(3)^2}{16}e_{\zeta(3)^2}(\sigma) - \frac{35\zeta(5)}{8}e_{\zeta(5)}(\sigma) + \frac{5\zeta(7)}{96}e_{\zeta(7)}(\sigma) - \frac{405}{64}e(\sigma),
\end{aligned}$$

where

$$\begin{aligned}
a &= \sqrt{\sigma}\Big(1 + \frac{1}{2\sigma} - \frac{1}{8\sigma^2} + \frac{1}{16\sigma^3} - \frac{5}{128\sigma^4} + \frac{7}{256\sigma^5} - \frac{21}{1024\sigma^6} + \frac{33}{2048\sigma^7} - \frac{429}{32768\sigma^8}\\
&\quad + \frac{715}{65536\sigma^9}\Big),\\
b &= \frac{1}{\sqrt{\sigma}}\Big(1 - \frac{1}{2\sigma} + \frac{3}{8\sigma^2} - \frac{5}{16\sigma^3} + \frac{35}{128\sigma^4} - \frac{63}{256\sigma^5} + \frac{231}{1024\sigma^6} - \frac{429}{2048\sigma^7}\Big),\\
c_{\zeta(3)} &= \sigma^{3/2}\Big(1 + \frac{3}{2\sigma} + \frac{3}{8\sigma^2} - \frac{1}{16\sigma^3} + \frac{3}{128\sigma^4} - \frac{3}{256\sigma^5} + \frac{7}{1024\sigma^6} - \frac{9}{2048\sigma^7} + \frac{99}{32768\sigma^8}\Big),\\
c &= \frac{1}{\sigma^{3/2}}\Big(1 - \frac{3}{2\sigma} + \frac{15}{8\sigma^2} - \frac{35}{16\sigma^3} + \frac{315}{128\sigma^4} - \frac{693}{256\sigma^5}\Big),\\
d_{\zeta(3)} &= \sqrt{\sigma}\Big(1 + \frac{1}{2\sigma} - \frac{1}{8\sigma^2} + \frac{1}{16\sigma^3} - \frac{5}{128\sigma^4} + \frac{7}{256\sigma^5} - \frac{21}{1024\sigma^6}\Big),\\
d_{\zeta(5)} &= \sigma^{5/2}\Big(1 + \frac{5}{2\sigma} + \frac{15}{8\sigma^2} + \frac{5}{16\sigma^3} - \frac{5}{128\sigma^4} + \frac{3}{256\sigma^5} - \frac{5}{1024\sigma^6} + \frac{5}{2048\sigma^7} - \frac{45}{32768\sigma^8}\Big),\\
d &= \frac{1}{\sigma^{5/2}}\Big(1 - \frac{5}{2\sigma} + \frac{35}{8\sigma^2} - \frac{105}{16\sigma^3}\Big),\\
e_{\zeta(3)} &= \frac{1}{\sqrt{\sigma}}\Big(1 - \frac{1}{2\sigma} + \frac{3}{8\sigma^2} - \frac{5}{16\sigma^3} + \frac{35}{128\sigma^4}\Big),\\
e_{\zeta(3)^2} &= \sigma^{5/2}\Big(1 + \frac{5}{2\sigma} + \frac{15}{8\sigma^2} + \frac{5}{16\sigma^3} - \frac{5}{128\sigma^4} + \frac{3}{256\sigma^5} - \frac{5}{1024\sigma^6} + \frac{5}{2048\sigma^7}\Big),
\end{aligned}$$

$$
\begin{aligned}
e_{\zeta(5)} &= \sigma^{3/2}\Big(1+\frac{3}{2\sigma}+\frac{3}{8\sigma^2}-\frac{1}{16\sigma^3}+\frac{3}{128\sigma^4}-\frac{3}{256\sigma^5}+\frac{7}{1024\sigma^6}\Big),\\
e_{\zeta(7)} &= \sigma^{7/2}\Big(1+\frac{7}{2\sigma}+\frac{35}{8\sigma^2}+\frac{35}{16\sigma^3}+\frac{35}{128\sigma^4}-\frac{7}{256\sigma^5}+\frac{7}{1024\sigma^6}-\frac{5}{2048\sigma^7}+\frac{35}{32768\sigma^8}\Big),\\
e &= \frac{1}{\sigma^{7/2}}\Big(1-\frac{7}{2\sigma}\Big).
\end{aligned}
$$

The coefficients, a, b, c_ζ, d_ζ and some of e_ζ begin to appear at the lower loops (1,2 and 3 loops). On the other hand, $c, d, e_{\zeta(3)}$ and e appear from $4, 6$ and 8 loops and therefore we need higher order calculation to accumulate enough perturbative information.

References

1 J. I. Kapusta, Finite temperature field theory, chap. 3 and 8, Cambridge monographs on mathematical physics (Cambridge Univ. Press, 1989).

2 R. Parwani, Phys. Rev. D45 (1992) 4695.

3 J. Frenkel, A. Saa and J. Taylor, Phys. Rev. D46 (1992) 3670.

4 P. Arnold and C. Zhai, Phys. Rev. D50 (1994) 7603.

5 R. Parwani and H. Singh, Phys. Rev. D51 (1995) 4518.

6 H. Yamada, Mod. Phys. Lett. A11 (1996) 1001; Mod. Phys. Lett. A11 (1996) 2793.

7 N. Mizutani and H. Yamada, *Modified Laplace transformation method and its application to the anharmonic oscillator*, physics/9702013, to appear in Int. J. Mod. Phys. A.

8 L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 3320.

9 A. Linde, Phys. Lett. B96 (1980) 289.

10 D. J. Gross, R. D. Pisarski and L. G. Yaffe, Rev. Mod. Phys. 53 (1981) 43.

Table Caption

Table 1 The coefficient functions are evaluated via the stationarity condition. The number in () represents the loop level at which the evaluation is made.

Figure Captions

Fig.1 Feynman diagrams dominant at each order in the small mass limit.

Fig.2 $\hat{M}_{dominant}(x)\beta^2/(8\zeta(3))$ at 2-loop level is plotted for $\beta = 1$ and $g = 0.05$.

coefficient						exact
a	1.12838(2)	1.03027(4)	1.01048(6)	1.00417(8)	1.00179(10)	1
b	0.75253(4)	0.892268(6)	0.948243(8)	0.974006(10)		1
$c_{\zeta(3)}$	0.56419(3)	0.96975(5)	0.99304(7)	0.997867(9)		1
c	0.300901(6)	0.549637(8)	0.718568(10)			1
$d_{\zeta(3)}$	1.12838(4)	1.03027(6)	1.01048(8)	1.00417(10)		1
$d_{\zeta(5)}$	1.06455(5)	1.00847(7)	1.00192(9)			1
d	0.08597(8)	0.25135(10)				1
$e_{\zeta(3)}$	0.75253(7)	0.892268(9)				1
$e_{\zeta(3)^2}$	1.06455(6)	1.00847(8)	1.00192(10)			1
$e_{\zeta(5)}$	0.56419(5)	0.96975(7)	0.99304(9)			1
$e_{\zeta(7)}$	0.423142(3)	0.619066(5)	0.983706(7)	0.997431(9)		1
e	0.0191(10)					1

Table 1